Linear Recursion

by
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1 Definition

Recursion is the repeated application of a formula or algorithm, where the result of the previous iteration is used as the argument for the current iteration.

Recursion requires an initial argument, denoted by a(1).

Mathematical Notation for $f(x)$:

$$\begin{cases} a(1) &= \text{initial argument} \\ a(n) &= f(a(n-1)) \end{cases}$$

2 Function Examples

The function $f(X) = bX + c$, where b and c are elements of the Integers $\mathbb{Z}$

The two functions $f(X) = X + 1$ and $f(X) = X - 1$ with a(1) = 0, generate the Positive Integers: 0,1,2,3,4,5,... and the Negative Integers: 0,-1,-2,-3,-4,-5,...

2.1 $f(X) = 2X + 1$

$$\begin{cases} a(1) &= 0 \\ a(n) &= 2(a(n-1)) + 1 \end{cases}$$

Step	Result
1	0
2	1
3	3
4	7
5	15
6	31

Closer examination of the sequence reveals a formula: $2^{(n-1)} - 1$

2.2 $f(X) = 3X + 1$

$$\begin{cases} a(1) & = & 0 \\ \\ a(n) & = & 3(a(n-1)) + 1 \end{cases}$$

Step	Result	Base 3
1	0	0
2	1	1
3	4	11
4	13	111
5	40	1111
6	121	11111

Closer examination of the sequence reveals a formula: $\dfrac{3^{(n-1)} - 1}{2}$

2.3 $f(X) = bX + 1$

$$\begin{cases} a(1) & = & 0 \\ \\ a(n) & = & b(a(n-1)) + 1 & = & \dfrac{b^{(n-1)} - 1}{b - 1} \end{cases}$$

The fact that ”b” emerges as the base in depicting the sequence as a series of repunits(1111...) is an interesting revelation, e.g. 7X + 1 results in 1111... in base 7, etc.

2.4 Other initial Values

When a(1) $\neq$ 0, the following sequence is generated:

$f(X) = 2X + 1$

$$\begin{cases} a(1) &= 5 \\ a(n) &= 2(a(n-1))+1 \end{cases}$$

Step	Result	Binary
1	5	101
2	11	1011
3	23	10111
4	47	101111
5	95	1011111
6	191	10111111

Upon examination, the original number, 5, in base 2, 101, is followed by repunits. This reveals a new formula for the recursion:

$$\begin{cases} a(1) &\neq 0 \\ a(n) &= b(a(n-1))+1 \quad = a(1)b^{(n-1)} + \dfrac{b^{(n-1)}-1}{b-1} \end{cases}$$

This holds for all bases > 1.

2.5 Other Values for c

The constant "c" gives a new surprise:

$f(X) = 2X + 2$

$$\begin{cases} a(1) &= 0 \\ a(n) &= 2(a(n-1))+2 \quad = \quad 2\left[\dfrac{2^{(n-1)}-1}{2-1}\right] \end{cases}$$

Step	Result	Binary
1	0	0
2	2	10
3	6	110
4	14	1110
5	30	11110
6	62	111110

When c is less than b, then the recursion results in repdigits (repeating digits) of c in base b.

$f(X) = 7X + 4$

Step	Result	Base 7
1	0	0
2	4	4
3	32	44
4	228	444
5	1600	4444
6	11204	44444

3 Summation

The general formula for $f(x) = bX + c$ Linear Recursion is:

$$\begin{cases} a(1) & \geq & 0 \\ a(n) & = & b(a(n-1)) + c \quad = \quad a(1)b^{(n-1)} + c\left[\dfrac{b^{(n-1)} - 1}{b - 1}\right] \end{cases}$$

That's All Folks!